

Thermodynamic resource theories, non-commutativity and maximum entropy principles

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With technology pushing into increasingly quantum-mechanical regimes, there are many motivations for us to deepen the connections between quantum information science and thermodynamics. One recent approach has been via resource-theoretic formulations, and in particular formulations that try to capture genuinely quantum mechanical properties within a thermodynamic context. Here we discuss how the set of free states of thermodynamic theories is strongly constrained in the presence of a microscopic conservation law; we show that for the simplest case of a Hamiltonian with no other conserved quantities the only non-trivial resource theories are either thermal operations or one of a range of coherence theories. We next discuss certain issues that arise when one introduces conserved charges into thermodynamic resource theories. Specifically we consider both commuting and non-commuting conserved charges, and highlight subtleties that exist between dynamical equilibration accounts, maximum entropy approaches and resource formulations. We conclude by constructing Landauer erasure bounds for these conserved charges. For this, we find trade-offs can exist between the energetic and other conserved charges erasure costs. It is found that the particular Landauer erasure bound we obtain is not sensitive to whether the charges commute or not.

I. INTRODUCTION

Quantum information science is driven both by practical considerations concerning what can and cannot be done with quantum states of matter, but also foundational considerations that seek to identify what makes quantum theory so distinct from the classical conception of the world.

A central focus, common to both practical and foundational research, is to understand the structure and uses of quantum entanglement. A key insight that gradually came to prominence is the idea that quantum entanglement can be defined by what it is not. Specifically, entanglement admits a *resource theory* formulation, in which entangled states are characterised as being those states that cannot be created under a particular set of quantum operations (namely local operations and classical communications). Such a formulation allows us to study the structure of entangled states, construct meaningful measures, and to develop an abstract language in which to reason precisely [1].

However, the basic machinery developed to study entanglement is also suited to shedding light on a much older subject – namely thermodynamics. Indeed quite recently a resource theory formulation of thermodynamics has been developed, and has seen an explosion of interest [2–22].

Such an approach offers a perspective on thermodynamics that naturally connects with contemporary quantum information theory techniques designed specifically to analyse intrinsically quantum-mechanical properties such as quantum coherence, entanglement and non-commutativity of quantum observables.

In what follows¹ we discuss some basic features of thermodynamic resource theories, and how quantum-mechanical aspects provide constraints on the type of theories that can arise. We also highlight some subtleties on the connection between resource theoretic approaches, dynamical equilibration and maximum entropy principles. Finally we consider how the information-theoretic task of Landauer erasure looks in the context of conserved, but non-commuting quantum charges.

The structure of this manuscript is as follows: the results of section II are based on results obtained previously in [14] and describe certain key aspects that are required to arrive at a non-trivial resource theory. Sections III, IV and V are adapted from lectures given at Imperial college [24] and work in [23]. These deal with the question of general conserved charges in thermodynamics (both commuting and non-commuting) and highlight certain subtleties that arise when one tries to formulate an account with non-commuting conserved charges. The final section analyses a concrete scenario in which such conserved charges are used to perform the basic task of Landauer erasure.

¹ The material presented here comprises of results that stem from a 2013 Master of Research at Imperial College London [23]. We have shared these results with people at various points since then, and given present interest in the community for such topics we hope that this discussion is of broader use.

II. NON-TRIVIAL RESOURCE THEORIES SUBJECT TO CONSERVATION PRINCIPLES

We shall describe how conservation principles can constrain the type of resource theories that can be constructed. Ultimately, we encode conservation laws in terms of unitary dynamics commuting with an observable, or set of observables.

We shall review some concepts that we first discussed in [14] in relation to the structure of the allowed free states of the theory, and whether a particular set of states “trivialises” the theory in a sense that we shall make precise. Here we shall present it from a slightly more general perspective that is relevant for what follows.

A. Resource theories and conservation laws.

A quantum resource theory is defined once we specify a set of free states and a set of free operations. In thermodynamics the first law of thermodynamics effectively states that energy is conserved microscopically. More explicitly, energy is conserved once all energy sources and sinks are included in the quantum formalism.² Given a composite system with total free Hamiltonian H_{tot} , the conservation principle can be encoded in terms of unitary dynamics obeying $[U, H_{\text{tot}}] = 0$, where H_{tot} is the Hamiltonian of the energetically isolated system. The set of free states is denoted by \mathcal{F} and it provides a backdrop for the theory. Specifically, we assume that \mathcal{F} is closed under an arbitrary number of finite tensor products, and we are free to experimentally prepare any particular state γ_B in \mathcal{F} . With this notion of free states of the theory, we now make minimal assumptions concerning how the free operations relate to the free states. We define a *H-conserving resource theory* by the set of free operations defined via

$$\mathcal{E}(\rho) = \text{Tr}_B[U(\rho \otimes \gamma_B)U^\dagger], \quad (1)$$

where $\gamma_B \in \mathcal{F}$, $[U, H_{\text{tot}}] = 0$ and $H_{\text{tot}} = H + H_B$ is the sum of system and bath Hamiltonians. This set of free operations is typically assumed, or shown, to be convex and we shall assume this for simplicity here.

It is important to emphasize that we do not place any further restrictions on the set of (global) unitaries, only the minimal assumption that they commute with the total Hamiltonian H_{tot} .

² Often for thermodynamic work protocols it is assumed that the experimenter has access to implicit mechanical systems in perfectly ordered states (such as a weight in a gravitational potential). In resource-theoretic accounts such systems must be modelled explicitly as quantum systems.

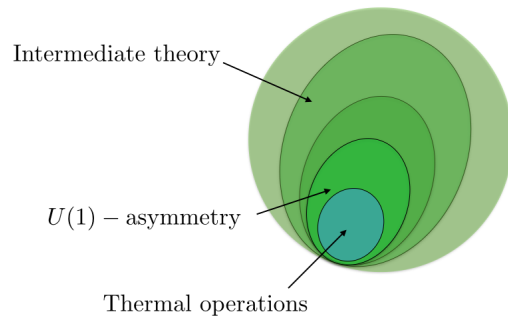


FIG. 1. **Non-trivial H -conserving resource theories.** Imposing only microscopic conservation on the underlying dynamics and the basic requirements of resource theories, we find that a set of non-trivial resource theories exist.

Given the above, we can now determine what non-trivial resource theories arise in terms of the set of free states we allow. It is important to emphasize that we draw a distinction between the term “ H -conserving theory”, and the particular case of thermodynamics. To say a theory is H -conserving makes no a priori assumption about the free states, it only fixes the underlying unitary dynamics. Given this caveat, we find the following:

Result 1 *If \mathcal{R} is a H -conserving resource theory with set of free states \mathcal{F} that contains the Gibbs state, then \mathcal{R} is either:*

1. *The resource formulation of thermodynamics (or a limiting case of zero or infinite temperature).*
2. *One of a family of coherence theories (in the sense of $U(1)$ -asymmetry [25]).*

This was proven in [14] in slightly less generality than stated here, where the goal was simply to emphasize that the possession of arbitrary amount of ordered energy *does not* trivialise thermal operations, as was previously claimed. We shall not reiterate the thermodynamic consequences of this here, but the interested reader can see [13, 14].

The above statement can be obtained from the following analysis of the set of free states \mathcal{F} . A key distinction between alternatives 1 and 2 in Result 1 is whether the set \mathcal{F} permits the arbitrary raising of a weight system or not. For simplicity we can phrase this in terms of the expected energy of the weight, as if the first moment of its Hamiltonian could be arbitrarily displaced upward. However, given the assumed existence of a ground state, it is clear that any fraction of the energy distribution can also be displaced upwards arbitrarily high and so any other notion of “work” will be trivialised as well.

We therefore wish to determine the set of quantum states \mathcal{F} that do not admit such arbitrary displacements. The answer to this was formulated in the 1970s, first from an algebraic perspective [26] and later by Lenard from more traditional finite-dimensional means [27]. The problem was previously phrased as determining the consistent families of states that are admissible. One consequence of this seminal work is that for finite-dimensional systems the Gibbs state is the only functional form (modulo limiting cases of ground states and “infinite temperature” systems) that is allowed, forcing the set \mathcal{F} to consist entirely of Gibbs states with a single shared temperature $\beta^{-1} = kT > 0$.³ In fact, the Gibbs state is the only *completely passive* state in the sense that arbitrarily many copies cannot raise a weight; more precisely, $\text{Tr}[H(U\rho^{\otimes n}U^\dagger - \rho^{\otimes n})] \geq 0$ for all $n \in \mathbb{N}$ and all unitaries U , including those that do not commute with H [26–28]. Moreover, if the previous inequality holds just for $n = 1$, then the state is simply called *passive*. Indeed [27], a state ρ is passive if and only if $[\rho, H] = 0$ and the eigenvalues of ρ monotonically decrease as a function of energy.⁴

Since a resource theory allows arbitrarily many copies of a free state, a H -conserving theory with non-trivial work processes can only have

$$\mathcal{F} \equiv \mathcal{F}_\beta = \{\gamma_B = \exp[-\beta H_B]/Z\}, \quad (2)$$

where $\dim(\gamma_B) < \infty$, $\text{Tr} \gamma_B = 1$, $\gamma_B \geq 0$. This in turn defines the set of thermal operations. However, it is of interest from a resource theory perspective to study what other non-trivial H -conserving resource theories are admissible.

A useful observation is that Gibbs states have two key properties: first, they have zero coherence; secondly, their spectra are monotonically decreasing with energy in just the right way (specifically, a Gibbsian profile). One can therefore consider dropping each of these components in turn and see what happens when we admit such states to \mathcal{F} .

Firstly, suppose \mathcal{F} contains only zero coherence quantum states, and in particular one with energy distribution that is not Gibbsian; then, by definition, it contains arbitrarily many copies of this state too. For simplicity we have assumed that the Gibbs state is available or can be constructed, and given this it can be shown (see [14]) that from the set \mathcal{F} we can arbitrarily approximate *every* zero coherence quantum state. The resultant resource theory is the $U(1)$ -asymmetry theory of quantum coherence with respect to H . The reason for this is that the covariant Stinespring dilation [29] implies that every quantum operation \mathcal{E} with the property that the group action e^{-itH}

commutes with the application of \mathcal{E} (namely “time covariance”, the defining property of the resource theory [25] of $U(1)$ -asymmetry) can be written in the form

$$\mathcal{E}(\rho) = \text{Tr}_B[U(\rho \otimes \sigma_B)U^\dagger] \quad (3)$$

where σ_B is a quantum state on an auxiliary system which is invariant under the unitary group action generated by its Hamiltonian, $[U, H + H_B] = 0$. Therefore the resulting resource theory \mathcal{R} must contain all free operations and states of the $U(1)$ -asymmetry theory as a subset. The assumption that \mathcal{F} contains no coherent states implies that all free operations are covariant, so we obtain the reverse inclusion which completes the proof. This result is particularly useful in showing that, even if we trivialise work processes, non-trivial coherence constraints still remain. In fact, such constraints go beyond what can be captured by free energy functions, as was shown in [13].

Finally, the last possibility is that \mathcal{F} contains (arbitrarily many copies of) coherent free states $\{\chi_k\}$. These states also allow unbounded raising of weights, however they also have coherent properties which extend the set of free states further. To deduce the structure of this set we resolve the states in mode operators as was done in [14] and originally in [30], and write $\chi_k = \sum_{\nu_k} \chi^{\nu_k}$ for $\nu_k \in \mathbb{Z}$ labelling the (one-dimensional irreps) of the group action. More intuitively: each state χ_k can be broken into terms that oscillate in time at definite frequencies labelled by ν_k – in other words we have performed a Fourier decomposition of the quantum state.

Since the group structure is so simple (abelian), the combination of multiple states has trivial composition and irrep labels simply add. Since \mathcal{F} is closed under tensor products, we deduce that we have a whole lattice of modes available,

$$\text{Modes}(\mathcal{F}) = \{n_k \nu_k : \text{for any } n_k \in \mathbb{N}\}. \quad (4)$$

However even this does not trivialise the resource theory. The reason is, again, that the imposition of H -conservation implies that no modes outside of this set can be created – in other words the resultant theory \mathcal{R} cannot trivialise the resource theory of coherence in the modes $\mathbb{Z} - \text{Modes}(\mathcal{F})$. We see that if $\text{Modes}(\mathcal{F}) = \{0\}$, then we recover the full resource theory of coherence with respect to the energy eigenbasis.

The main point of the above analysis is that the conservation principle applied to quantum resource theories requires us to determine consistent families of states with non-trivial, or interesting structure. In the next section we consider situations in which one not only has conservation of energy, but also of other observables, and show how this constrains the type of free states allowed.

³ For infinite dimensional systems such as quantum fields this notion is generalized to the set of KMS states and has a more rich and complex structure, which we entirely ignore here.

⁴ Note that since $[\rho, H] = 0$ the state is diagonal in the energy eigenbasis.

III. CONSERVED CHARGES, MAXIMUM ENTROPY PRINCIPLES AND FREE STATES

In the previous section we discussed how free states affect the types of resource theories one has in the presence of a conservation principle. From this perspective the Gibbs state is singled out as being special in terms of allowing a resource theory of irreversible thermodynamic processes. We would like to look more broadly to other thermodynamic quantities and see whether similar non-trivial constructions can be made.

In this regard, it is useful to discuss different ways in which the Gibbs state is special, and how these notions relate to one another. In particular the Gibbs state is singled out:

1. In terms of the fixed point of dynamical equilibration processes.
2. In terms of the maximum entropy state consistent with an energetic constraint.
3. In terms of complete passivity for the set of free states, as discussed in section II.

Given these different views we now consider whether one perspective can shed light or be a useful guide within another. We shall also describe how the three perspectives are connected, but will not delve into the vast literature on such topics. In what follows we shall consider the question of conserved charges in thermodynamics and pose the question: do there exist non-classical features that arise due to the thermodynamics of these charges being non-commuting?

A. The grand canonical ensemble

In traditional thermodynamics there are certain bread-and-butter statistical ensembles – the microcanonical, the canonical and the grand canonical. In the previous section we provided an account of how the canonical ensemble fits into the resource framework, and the role it plays there. However, exactly the same kind of analysis can be conducted for the grand canonical ensemble.

It is useful to run the above three perspectives in parallel in seeing how they describe the grand canonical ensemble. From a statistical perspective one considers a thermodynamic system with many microscopic degrees of freedom that is subject to two thermodynamic constraints [31]. The constraints allow the exchange of energy (in the form of heat) with the surroundings, and also allow the exchange of *particles* with the ambient surroundings, while assuming that both these quantities are conserved microscopically; the long-time dynamics and ergodicity can be used to deduce an equilibrium regime. The former constraint generates a notion of temperature $kT = \beta^{-1}$, while the latter a chemical potential μ , and the resulting equilibrium distribution is given by

$$\rho_{GC} = e^{-\beta(H-\mu N)} / Z, \quad (5)$$

where $Z = \text{Tr}[e^{-\beta(H-\mu N)}]$, and N is the particle number operator (and $[H, N] = 0$).

The maximum entropy principle [32–34] encodes the thermodynamic equilibration configuration in the statement that $\langle H \rangle = \text{constant}$ and $\langle N \rangle = \text{constant}$. The assumed ergodic dynamics is then encoded in the statement that all other degrees of freedom have been maximally mixed subject to the above two constraints. Or, put another way, any measured data should only reflect these constraints. Following the seminal work of Jaynes [34], Shore and Johnson [35] proved that under very natural axioms on the handling of measured data, there is essentially a unique way to present such a constraint problem, namely one obtains the Gibbs distribution as the unique solution to

$$\begin{aligned} &\text{Maximize } H(\mathbf{p}) = -\sum_i p_i \log p_i \\ &\text{subject to: } \langle H \rangle = \bar{u}, \text{ and } \langle N \rangle = \bar{n}, \end{aligned}$$

where \bar{u} and \bar{n} are the average energy and particle number of the system respectively. More recently, Caticha [36] has shown how this principle can be generalized further to include more general notions of constraints, and how one can interpolate between Bayesian updating and the original maximum entropy construction. Quantum-mechanically, the max entropy principle is traditionally expressed as the following problem:

$$\begin{aligned} &\text{Maximize } S(\rho) = -\text{Tr}[\rho \log \rho] \\ &\text{subject to: } \text{Tr}[\rho H] = \bar{u}, \text{ and } \text{Tr}[\rho N] = \bar{n}, \end{aligned}$$

that gives as a unique solution Eq. (5).

Finally, from the perspective of the family of consistent free states, the grand canonical is special from the perspective of being (H, N) -completely passive. In other words, we impose the condition that not only the free states cannot be used to raise a weight to an arbitrary height (complete passivity of energy), but moreover we cannot raise the particle number of an ordered system to an arbitrary value (complete passivity of particle number). It is then clear that imposing passivity in terms of H and N firstly implies at the single copy level that any (H, N) -passive state ρ_* satisfies $[\rho_*, H] = [\rho_*, N] = 0$, so that ρ is (block)-diagonal in the common eigenbasis of H and N . Moreover, the spectrum of ρ_* , when marginalised over either H or N degrees of freedom, must be a monotonically decreasing function of the eigenvalues of H and N . Over arbitrarily many copies this becomes complete passivity and we have that these marginals must be Gibbsian distributed. This shows that the state $\rho_* = \exp[-\lambda_1 H - \lambda_2 N] / Z$, where Z is the normalization of the state, is completely passive.⁵ Indeed, there has already been resource theoretic work exploring this

⁵ We implicitly assumed structural stability, which demands that an infinitesimally small perturbation of degenerate energy levels also results in a passive state. In addition, subtleties exist regarding the independence of H and N degrees of freedom.

particular choice of free states [37, 38] and we refer the reader to them.

B. Information-theoretic entropy and traditional accounts of thermodynamic equilibrium.

We now make a few remarks that will be of relevance later. The maximum entropy approach just described revolves around an *information-theoretic* entropy function, however it is important to make contact with *thermodynamic* entropy.

What follows does not apply to single-shot scenarios, but instead to the asymptotic, equilibrium regime of thermodynamics. The relevance is that the set of free states \mathcal{F} of the resource theory include such states – arbitrarily large and in equilibrium in the sense described in the previous section. Indeed, it is well-established (for example see the elegant axiomatization of thermodynamics [39]) that a unique thermodynamic entropy function exists only in the particular regime of asymptotic, equilibrium thermodynamics. For such a regime this monotonically non-decreasing entropy function is uniquely specified (modulo “affine transformations” [39]).

How can we reconcile such a non-decreasing function with maximum entropy principles? For this, we can consider the more general situation of arbitrary quantum states, denoted by density operators ρ, σ, \dots , and consider the experimental situation of us only having access to a restricted set of physical parameters

$$\mathcal{L} = \{\langle X_1 \rangle, \langle X_2 \rangle, \dots, \langle X_N \rangle\}. \quad (6)$$

Any testable prediction about the system must be expressed using this set. One might expect that the thermodynamic entropy is consistent with the maximum entropy principle of not claiming knowledge we do not have.

Given a specific set of data for these parameters we can apply maximum entropy and construct the optimal state for our predictions. The set of states resulting from all possible values of the parameters in Eq. (6) is called the Gibbs manifold \mathcal{M} of states for \mathcal{L} . Furthermore, given any state ρ with expectation values $\{\langle X_k \rangle_\rho\}$ we can define a mapping/projection of ρ into the Gibbs manifold \mathcal{M} via the maximum entropy principle. One computes the state σ with the *same* expectation values $\{\langle X_k \rangle_\sigma = \langle X_k \rangle_\rho\}$, but with maximal von Neumann entropy $S[\sigma]$. We can write this simply as $\rho \rightarrow \pi(\rho) \equiv \sigma$ and can call σ a (X_1, \dots, X_N) -bath.

One approach to defining the thermodynamic entropy with respect to the description \mathcal{L} as $S_{\text{thermo}}[\langle X_1 \rangle, \dots, \langle X_N \rangle]$ is as follows:

$$S_{\text{thermo}}[\langle X_1 \rangle, \dots, \langle X_N \rangle] := -\text{Tr}[\sigma \log \sigma] \quad (7)$$

where $\sigma \in \mathcal{M}$ is the maximum entropy state constructed from the values $\{\langle X_1 \rangle, \dots, \langle X_N \rangle\}$. This might seem like it has little content, but it has important consequences. For a start, it is not simply the von Neumann entropy

of the quantum state, secondly it is consistent with a dynamical equilibration account as we shall explain.

A general form of the thermodynamic increase of entropy follows from (a) this definition and (b) the unitary invariance of the von Neumann entropy. The argument goes as follows. Consider having prepared an initial state $\rho \in \mathcal{M}$ for the system. Now suppose it is subjected to some unitary process so that $\rho \rightarrow U\rho U^\dagger \equiv \rho(t)$. Let’s compare the thermodynamic entropy with respect to \mathcal{L} at the start and at the end. We know initially that $S_{\text{thermo}}(0) = S_{\text{thermo}}[\langle X_1 \rangle, \dots, \langle X_N \rangle] = -\text{Tr}[\rho \log \rho]$. The von Neumann entropy is unitarily invariant so $-\text{Tr}[\rho(t) \log \rho(t)] = -\text{Tr}[\rho \log \rho]$, however $-\text{Tr}[\rho(t) \log \rho(t)]$ is *not* the thermodynamic entropy at the final time. By definition, the thermodynamic entropy at the later time is $S_{\text{thermo}}(t) = -\text{Tr}[\pi(\rho(t)) \log \pi(\rho(t))]$. However, the state $\pi(\rho(t))$ has von Neumann entropy greater than or equal to $\rho(t)$. Thus

$$\begin{aligned} S_{\text{thermo}}(t) &= -\text{Tr}[\pi(\rho(t)) \log \pi(\rho(t))] \\ &\geq -\text{Tr}[\rho(t) \log \rho(t)] = S_{\text{thermo}}(0) \end{aligned} \quad (8)$$

This establishes the non-decreasing aspect of the thermodynamic equilibrium entropy as a function of time from a maximum entropy perspective (see [40] for a classical discussion).

A key point should be emphasized in this regard. The thermodynamic entropy S_{thermo} as defined is manifestly a function of the *thermodynamic state* only, and does not carry a history of the dynamics, whereas if we had used simply $S(U\rho U^\dagger)$ we would be putting in our knowledge of the prior evolution, given by U . From the dynamical perspective, it is precisely this that is “washed away” in the equilibration process.

The maximum entropy perspective will prove useful in the next section, where we discuss the construction of free states of a resource formulation containing conserved non-commuting charges.

IV. NON-COMMUTING CONSERVED CHARGES

A central component of thermodynamics is irreversibility in physical systems as they evolve in time to equilibrium. Moreover, the thermodynamic equilibrium quantities of interest are typically assumed to be stable in time and in traditional accounts admit equilibrium values in terms of conserved quantities.

A good deal of attention has been devoted lately to the issue of identifying “genuinely quantum mechanical” features in non-macroscopic quantum systems. The most obvious candidates to consider are entanglement in thermodynamics, coherence [41] and issues related to non-commutativity. A good deal of work has been done on the relations of entanglement and thermodynamics [42–46], and more recently initial results in the resource formulation of thermodynamics have begun to analyse in

a rigorous way the role that coherence plays in thermodynamics. In particular it was shown in [13] that it is *impossible* for free energy functions to capture the thermodynamic ordering of coherent quantum states under thermal operations. This was complemented by work that tried to establish the features not captured by free energies [9, 11, 12, 14, 19]. Much still remains to be done in this regard.

At present it is unclear to what extent non-commutativity impacts thermal resource theories. Indeed, at first glance the above considerations on conserved quantities might seem to prohibit the inclusion of non-commutativity into this setting. However, one can simply have non-commutativity among *multiple* conserved charges. This suggests an extension of the H -conserving resource theory of Eq. (1) to the non-commuting case as follows (stated for two charges for simplicity): a (H, A, B) -conserving theory is defined as

$$\mathcal{E}(\rho) = \text{Tr}_B[U(\rho \otimes \gamma_B)U^\dagger], \quad (9)$$

where γ_B is any free state in the set \mathcal{F} of generalized bath states, and $[U, H_{\text{tot}}] = [U, A_{\text{tot}}] = [U, B_{\text{tot}}] = 0$. Here A_{tot} and B_{tot} denote the sum of the charge observables on the primary system and their corresponding observables on the generalized bath system. here $A_{\text{tot}} = A + A_B$, and similarly for B . We are in particular interested in the case where $[A, B] \neq 0$.

Researchers in the quantum information community have already considered information-theoretic aspects of spin baths in thermodynamics, for example [47], and more recently [48]. However here we would like to address two different questions:

1. What constraints are put on free bath states \mathcal{F} in the case of conserved non-commuting charges?
2. Is there concordance between the maximum entropy principle, resource formulations, and the notion of dynamical equilibration in the presence of non-commuting charges?

We would therefore like to consider the set of free states that provide some non-trivial theory when we impose a notion of conservation on non-commuting observables. The details of this analysis motivates the second question posed, and therefore we shall consider both together in the next section.

A. Resource theories for conserved non-commuting charges.

The reason we have drawn attention to maximum entropy principles in this discussion is because of the observation that: *maximum entropy constructions work equally well for commuting or non-commuting observables*. Given constraints on the expectations of quantum observables $\{Q_1, Q_2, \dots\}$, the maximum entropy state is again unique, and of the same Gibbsian form

$\rho_\star = \exp[-\lambda_1 Q_1 - \lambda_2 Q_2 \dots]/Z$. It therefore takes the same functional form as the bath states for the canonical and grand canonical scenarios.

Given that the Gibbsian form arises under maximum entropy constructions, one can ask whether it is possible to have concordance with dynamical equilibration and complete passivity in the non-commuting regime.

It turns out the case of a set of conserved charges $\mathcal{Q} = \{Q_1, Q_2, \dots\}$, some of which are non-commuting, has interesting features in this regard. For simplicity we can consider the example of $\mathcal{Q} = \{X, Y\}$, where X and Y are Pauli operators in an effective qubit subsystem. Aside from the energy spectrum constraints (which are independent due to commutativity), the maximum entropy state on this effective qubit subsystem takes the form $\rho_\star = \exp[-\lambda_1 X - \lambda_2 Y]/Z$, under constraints on the expectation values of X and Y . A dynamical account of equilibration would involve a non-equilibrium state, subject to these constraints, that evolves under some dynamics to ρ_\star (see Fig. 2).

However such an equilibration process cannot coincide with the maximum entropy construction. The reason is that for this particular model, such a map turns out to be a positive, but not *completely positive* quantum map, in the sense that if applied to a subsystem of an arbitrary entangled state it can generate negative probabilities.⁶ Therefore, such equilibration to the maximum entropy state ρ_\star cannot occur as described.⁷

⁶ This is seen by computing $\tilde{\rho} = \mathcal{E} \otimes \text{id}(|\Omega\rangle\langle\Omega|)$ on the pure bipartite, maximally entangled state given by $|\Omega\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

Since $|\Omega\rangle\langle\Omega|$ can be written as $\frac{1}{4}(\mathbb{1} + XX)(\mathbb{1} + ZZ) = \frac{1}{4}(\mathbb{1} + XX + ZZ - YY)$, one finds that $\tilde{\rho} = \frac{1}{4}(\mathbb{1} + XX - YY)$ which has eigenvalues $\{\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\}$, and therefore does not correspond to a physically allowed quantum state on the global system.

⁷ A similar situation can occur even for the case of A and B commuting, in $d \geq 3$.

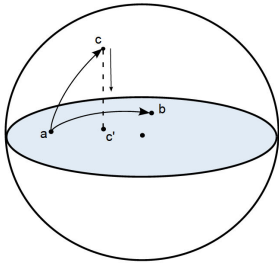


FIG. 2. **Non-commuting bath degrees of freedom.** The set of Gibbs states (shaded disk) of a constraint problem in terms of Pauli X and Pauli Y for a qubit degree of freedom (for simplicity we assume $H \propto \mathbb{1}$). The transformations $a \rightarrow b$ and $a \rightarrow c$ are both unitary, but the former corresponds to a reversible change of external parameters that keeps the state in the shaded disk, while the latter moves off the Gibbs manifold. The maximum entropy projection to the state $c' = \pi(c)$ does not correspond to a physically realisable equilibration process. The maximum entropy state is Gibbsian and of the form $\exp[-\lambda_1 X - \lambda_2 Y]/Z$, that can be written in a frame independent form $\exp[-\boldsymbol{\lambda} \cdot \boldsymbol{\sigma}]/Z$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, 0)$ denotes a particular physically defined axis of the problem. The case $\boldsymbol{\lambda} = \mathbf{0}$ corresponds to the case where there is direct agreement between the three equilibrium perspectives. In contrast, for non-zero $\boldsymbol{\lambda}$ other considerations must be taken into account.

One also obtains subtleties when we consider the passivity construction of free resource states for non-commuting conserved charges. Our earlier discussion of H -conservation shows that one must face the issue of how to construct a set of free states \mathcal{F} , which is consistent in the sense of not trivialising the degrees of freedom one wishes to consider, while simultaneously respecting the conservation laws in place. It turns out for a non-commuting case that this problem can be viewed in a number of ways, which we discuss here and in the next subsection.

We can begin by considering the scenario, where one wishes to define a bath state ρ_* to be (A, B) -completely passive in the sense that given many copies of the state, we cannot increase the expectation value of either A or B without bound. We now see what constraints are present in the case of $[A, B] \neq 0$. In particular, (A, B) -complete passivity implies that ρ_* is separately A -passive and B -passive. As already discussed above, this implies $[\rho_*, A] = [\rho_*, B] = 0$. However these conditions are satisfied if and only if

$$\rho_* = \sum_k c_k C_k, \text{ with } c_k \in \mathbb{R}, \quad (10)$$

for some collection of observables $C_k \in \text{Com}(A) \cap \text{Com}(B)$, where $\text{Com}(X)$ is the commutant of the operator X . The significance of this is that any (A, B) -passive state *cannot* contain any component of A or B in it. To explain what this means, we can consider for simplicity the case of a trivial Hamiltonian, and where the only operator that commutes with A and B is one proportional to the identity. For this

the only passive or completely-passive state with respect to (A, B) is the maximally mixed state $\frac{1}{d}\mathbb{1}$. In particular it is *impossible* to reproduce the maximum entropy Gibbsian distribution in the A and B degrees of freedom. This is not to say that a resource theory is impossible, instead the bath states act as random noise in the non-commuting charge degrees of freedom. If we include non-trivial Hamiltonians and/or other charges that commute with A and B then the free states are subject to a Gibbsian distribution in energy, however we still get a disconnection with maximum entropy constructions in the A and B degrees of freedom.⁸

Therefore we have shown that a disconnection occurs between maximum entropy principles and both the dynamical and resource-theoretic formulations. In the next section we investigate this further and discuss alternative models. In the final section we shall illustrate some of these concepts with the concrete example of Landauer erasure using different types of charge baths.

B. Weaker constraints, physical considerations and Gibbs free states.

The preceding analysis showed that a direct application of passivity notions lead to strong constraints on the A and B degrees of freedom in the free states of the resource theory, which differ from the above maximum entropy state. It is therefore of interest to elaborate further on the link between resource theories and the maximum entropy construction. For the sake of simplicity we illustrate it for the case of the canonical Gibbs state. For a Hamiltonian H it is straightforward to show that one can introduce a notion of free energy via $S(\rho||\gamma) := \beta(F(\rho) - F(\gamma))$, where γ is the Gibbs state with a temperature given by $kT = \beta^{-1}$. Expanding out the left-hand-side, it is immediately found that $F[\rho] = \text{Tr}[H\rho] - \beta^{-1}S(\rho)$ is the relevant free energy function. However, since $S(\rho||\sigma) \geq 0$ with equality if and only if $\rho = \sigma$, we recover the age-old result that the Gibbs state minimizes the free energy. Put another way, the expression for $F[\rho]$ immediately implies that the Gibbs state must therefore uniquely *maximize* entropy for a fixed energy expectation value $\text{Tr}[H\rho] = \bar{u}$. Moreover, the connection with passivity follows if we consider a unitary transformation that is allowed to change the energy expectation value. From the unitary invariance of the von-Neumann entropy, such unitary gives a change in free energy equal to $\Delta F = \beta(\text{Tr}[HU\rho U^\dagger] - \text{Tr}[H\rho])$; however since the Gibbs state minimizes the free energy, it follows that no unitary (more generally, no constant

⁸ Note that if $\{C_k\}$ together with A and B form an orthogonal set of operators, then we always have $\langle A \rangle = \langle B \rangle = 0$, otherwise we can have non-zero expectation values. However in both cases there is generally a disconnection with the full set of maximum entropy Gibbs states in A and B .

entropy transformation) can decrease the energy expectation and therefore the state is passive [49]. Moreover, since free energy is an *additive* function, precisely the same argument holds for arbitrarily many copies of $\rho^{\otimes n}$ and thus one deduces complete passivity in the same manner.

For the case of non-commuting charges (and $H \propto \mathbb{1}$ for simplicity) one can turn things around and deduce a weaker condition than in the previous section which, instead of yielding strong constraints on the A - B degrees of freedom, gives a Gibbsian form. Specifically if we consider $S(\rho||\rho_\star)$ where $\rho_\star = \exp[-\lambda_1 A - \lambda_2 B]/Z$, and make use of $S(\rho||\rho_\star) \geq 0$ then we see that the maximum entropy state is only passive in the sense that $\lambda_1 \text{Tr}[A\rho] + \lambda_2 \text{Tr}[B\rho] := \text{Tr}[\mathcal{O}(\lambda_1, \lambda_2)\rho]$ cannot be decreased in a constant entropy transformation, such as over the set of all unitaries, for fixed λ_1 and λ_2 . Here the maximum entropy principle is therefore associated to a weaker form of passivity, where a state is only required to be $\mathcal{O}(\lambda_1, \lambda_2)$ -completely passive, instead of (A, B) -completely passive. It is only if $[A, B] = 0$ that this can be further split into *independent* variations in $\langle A \rangle$ and $\langle B \rangle$, and the two approaches coincide.

Observe that one can interpret λ_1 and λ_2 as either Lagrange multipliers in an extremum problem, or as physical constraint parameters that are defined from the physics. There are good reasons why it may be sensible to frame the problem in this way. For example, one can view λ_1 and λ_2 as local field strengths within our Hamiltonian that couple to these non-commuting observables. Now it is well-established that the low temperature states of interacting spin systems (such as Ising models with transverse magnetic fields) display thermodynamic phase transitions depending on the particular external field parameters, and are intrinsically quantum-mechanical in origin. We may argue that a resource theory formulation involving non-commuting conserved charges benefits from starting with this weakened form of passivity, computing its properties, and then ascertaining if subsequent variation of the parameters displays discontinuities in these properties. More generally, the weaker form of $\mathcal{O}(\lambda_1, \lambda_2)$ -passivity can be viewed as the original notion applied to a fixed “direction” that is singled out as special.

The consideration of the actual physics of the problem leads to another perspective. If one has non-commuting conserved charges, then these correspond to physical observables with a non-trivial algebraic structure, for example the algebra of spin observables. An alternative approach to constructing a resource theory using this full structure would be to consider the resultant group G generated by the algebra of the charges as providing the ultimate constraint on the resources and quantum operations allowed. However this coincides with the general theory of asymmetry as developed in recent years [25, 50–52]. For the case of spin observables the free states are simply states symmetric under the group action $U(g)\rho_\star U(g)^\dagger = \rho_\star$ for all $g \in SU(2)$. This is consis-

tent with the strong form of passivity in the A - B degrees of freedom, in which the free state was constructed solely from observables in $\text{Com}(A) \cap \text{Com}(B)$ – namely only those observables that are invariant under the group action of G . This perspective also provides insight into the weaker notion of passivity in which one obtains a Gibbsian free state in A and B – one can understand this as a physical breaking of the symmetry group due to a preferred external constraints specified by λ_1 , and λ_2 (see Fig. 2). This reduces the full group to a commuting subgroup, $U(1)$.

V. LANDAUER ERASURE WITH CONSERVED QUANTUM CHARGE BATH STATES

In this section we use Landauer erasure as a context in which to explore the thermodynamics of conserved charges, and to see if non-commutativity can give rise to novel features above and beyond what we have already discussed. In order to determine this, it is first necessary to consider the commuting charge case, and compare with the results for the more subtle non-commuting case.

A. Information erasure in charge degrees of freedom

Landauer erasure is one of the central contexts in which thermodynamics and information theory interact non-trivially. In its simplest form it says that in order to erase a single bit of information it is necessary to dissipate a total amount of heat of $kT \ln 2$. Here heat flow is quantified in terms of the expectation values $\text{Tr}[H\Delta\rho]$, if H is the Hamiltonian of the system and $\Delta\rho$ is the change in the state during thermalisation. For pedagogical clarity we shall not consider issues of deterministic, or ϵ -certain quantities but restrict ourselves to expectation values. This is still valid in the single-shot, resource theoretic context, however with the caveat that these macroscopic, asymptotic quantities (such as free energy) are necessary but not sufficient to characterise the non-asymptotic regimes [7].

The usual way of describing erasure of energy levels by means of a time-varying Hamiltonian of the system cannot be simply carried over to the erasure via charges in a resource theory picture. Erasure is a process consisting of quasi-statically increasing the energy gap by changing the Hamiltonian H in time, while keeping the system in contact with a heat bath. When the gap is very large (formally, infinity) all population is transferred to the ground state. Once the bath is disconnected, we can freely reduce the gap by transforming the Hamiltonian back to what it was at the initial time.

This becomes more subtle in the context of conserved charges, because there is no guarantee, a priori, that a continuous variation of the charge degree of freedom is either possible, or easily achieved. Moreover, within the

simplest resource-theoretic framework neither the Hamiltonians, nor the charge observables are allowed to vary in time. A simple way around this that leads to thermodynamic results expressed in terms of expectation values is to use the approach of [53]. For our context, we would therefore consider a sequence of free bath states that have charge degrees of freedom with varying spectral gaps. The system with fixed Hamiltonian and fixed charge observable would then sequentially interact with each bath, and this interaction can induce the desired quasi-static transformation.

Another approach, developed by Vaccaro and Barnett, also provides a neat protocol for spin bath erasure in a formal and precise way [47]. We now adapt their results to our thermodynamic scenario. This firstly allows us to (formally) provide a simple bridge between resource theories and more conventional accounting; secondly, from a physical perspective it is a protocol well-suited to erasure of information encoded in observables different from energy.

Consider a two level system, where $|0\rangle$ and $|1\rangle$ are eigenstates of some conserved charge Q with eigenvalues $q^{(0)} = q^{(1)}$. This is the system we want to erase. We will denote by \mathcal{H} the Hilbert space generated by the span of $|0\rangle$ and $|1\rangle$. Assume the initial state on \mathcal{H} to be erased is

$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|. \quad (11)$$

Traditional erasure protocols can be formalised as the sequential repetition of two operations:

Op.1: Level rising. The observable Q , is mapped to \tilde{Q} so that if

$$Q|i\rangle = q_i|i\rangle, \quad \tilde{Q}|\tilde{i}\rangle = \tilde{q}_i|\tilde{i}\rangle, \quad i = 0, 1,$$

the level splitting increases:

$$\delta Q = |\tilde{q}_1 - \tilde{q}_0| - |q_1 - q_0| > 0.$$

At the same time, the state $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ is mapped to $\tilde{\rho} = p|\tilde{0}\rangle\langle \tilde{0}| + (1-p)|\tilde{1}\rangle\langle \tilde{1}|$, with same spectrum but new eigenstates. For the sake of simplicity it is often assumed that we can raise and lower levels in an effectively continuous fashion, i.e. $\delta Q \alpha \ll 1$, if α is the inverse temperature of the Q -bath. However there are also situations in which discreteness cannot be ignored for physical reasons, such as minimal units of angular momentum.

Op.2: Thermalisation. After level rising, interaction of the system with a bath is allowed for a long enough time. This changes the spectrum of ρ , $(p, 1-p)$, to the correspondent Boltzmann distribution with respect to the current Hamiltonian, but leaves the eigenstates unchanged.

We can simply take each step separately, and discuss what each process demands from a resource theory perspective. Let us begin with manipulating the level structure, Op. 1. This requires work in the traditional accounting, so it cannot be a free operation in the resource theory. In fact, it can be formally modelled explicitly with the introduction of auxiliary degrees of freedom in pure states. These are manipulated to achieve the desired task, and we can compute the costs for each step of the protocol.

As a first step we add a new quantum state extending the system Hilbert space from \mathcal{H} to $\mathcal{H} \otimes \mathcal{H}_q$. For the system \mathcal{H}_q we can define an observable – call it Q_1 – which is microscopically conserved in combination with Q and gives $\tilde{Q} = Q_1 + Q$. The system in \mathcal{H}_q is assumed to be in the eigenstate $|0\rangle\langle 0|$ (eigenstate of Q_1), so that the state of the composite system is $\rho \otimes |0\rangle\langle 0|$. At this point one performs a CNOT with the auxiliary system to obtain

$$\tilde{\rho} = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11| \quad (12)$$

This unitary does not conserve \tilde{Q} and in fact requires one to draw charge (e.g. work, if $Q = H$) from a battery system. This battery system can be thought of as a ladder system, the analogue of the weight in the case $Q \equiv H$. For example in the case of $Q = J_z$, the angular momentum about some axis z , the battery provides “ordered” angular momentum instead of “ordered” energy. The average cost of performing this can be computed to be $\Delta Q = \text{Tr}[\tilde{Q}(\tilde{\rho} - \rho \otimes |0\rangle\langle 0|)]$, where $U = \text{CNOT}$.

The repetition of the protocol above essentially defines a one-parameter family of states $\rho(k)$ and observables $Q(k)$ on $\mathcal{H} \otimes \mathcal{H}_q^{\otimes k}$ such that the spectrum $(q_k^{(0)}, q_k^{(1)})$ of the observable $Q(k)$ is sequentially mapped into the spectrum $(q_{k+1}^{(0)}, q_{k+1}^{(1)})$ of $Q(k+1)$, in such a way that the difference $q_k^{(1)} - q_k^{(0)}$ increases with k . Moreover, the two eigenstates of $\rho(k)$ associated to non-zero eigenvalues $(p, 1-p)$ are eigenstates of $Q(k)$ with eigenvalues $q_k^{(1)}$, $q_k^{(0)}$, whose difference increases with k :

$$\rho(k) = p|q_k^{(0)}\rangle\langle q_k^{(0)}| + (1-p)|q_k^{(1)}\rangle\langle q_k^{(1)}|, \quad (13)$$

where

$$Q(k)|q_k^{(0)}\rangle = q_k^{(0)}|q_k^{(0)}\rangle, \quad (14)$$

$$Q(k)|q_k^{(1)}\rangle = q_k^{(1)}|q_k^{(1)}\rangle, \quad (15)$$

$$q_{k+1}^{(1)} - q_{k+1}^{(0)} > q_k^{(1)} - q_k^{(0)} \quad \forall k. \quad (16)$$

But this is exactly what we want to achieve with Op.1 and so it demonstrates that one can induce effective level-raising within the framework of conserved charges.

On the other hand, Op. 2 is thermalisation of the system with some general bath system. Mapping a state to the correspondent Gibbs state is always a free operation of the theory, but it is not necessarily what we want to do here. In fact, despite the introduction of virtual

subsystems \mathcal{H}_q to model level splitting, we want to thermalise as if it always was a qubit. Specifically, if we thermalise with respect to $Q(k)$ then unwanted levels would get populated. To tackle this we note that thermalising with respect to a subset of the energy levels is also an allowed operation in the resource theory. Therefore one can thermalise as follows:

$$\rho(k) \mapsto \frac{e^{-\beta q_k^{(0)}}}{Z} |q_k^{(0)}\rangle\langle q_k^{(0)}| + \frac{e^{-\beta q_k^{(1)}}}{Z} |q_k^{(1)}\rangle\langle q_k^{(1)}|, \quad (17)$$

where $Z = e^{-\beta q_k^{(0)}} + e^{-\beta q_k^{(1)}}$. This shows that also Op.2 can be formally induced within the resource framework. Note that the original Vaccaro-Barnett protocol [47] is performed using a spin bath and the level splitting is modified as in the discussion above by introducing individual spin systems. In practice, such a thermalisation step would require fine control of the system-bath interaction to prevent unwanted levels becoming populated (e.g. after the first step one does not want $|01\rangle$ to become populated).

B. Trade-offs in erasure with multiple commuting conserved charges

We can now consider what happens when one has access to different types of free states, and build a protocol from the core primitives of induced level splitting and thermalisation (namely Op. 1 and Op. 2 respectively) for multiple conserved charges.

The first, and simplest is the thermodynamic case of a system with Hamiltonian H , and a single (microscopically) conserved charge Q , which may for example be taken to be the number operator. As discussed the free states of the resultant theory are of the form

$$\gamma_B = \frac{e^{-\beta H_B - \alpha Q_B}}{Z}, \quad (18)$$

where H_B and Q_B are observables on the bath's Hilbert space physically corresponding to H and Q in the system, in the sense that the microscopic conservation laws read $[U, H + H_B] = [U, Q + Q_B] = 0$ if U is a joint unitary coupling system and bath (see Eq. (9)). β and α are fixed inverse temperatures. Moreover, since we assume $[H_B, Q_B] = 0$, one can formally factor the quantum state, and view it as if we have two independent baths at our disposal (a thermal bath and a “ Q -bath”) and we are able to put the system in contact with each of them separately.

Consider for simplicity the system Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, spanned by the eigenstates of two commuting observables $H \otimes \mathbb{1}$ and $\mathbb{1} \otimes Q$. Define the states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|hq\rangle := |h\rangle \otimes |q\rangle$ and $|h\rangle, |q\rangle$ are eigenstates of H and Q , respectively. Assume these four states to be initially degenerate in energy H and charge Q .

Erasure consists of mapping an initially maximally mixed state with degenerate Hamiltonian into a “default”

pure state (with the same Hamiltonian), usually with the aid of a thermal bath. A standard calculation shows that if we use a thermal bath, the minimum heat dissipated in the process amounts to $kT \log 2$ for a qubit system. Typically Landauer's principle is considered as a fundamental lower bound between the process of information erasure and dissipation of *energy* into a bath (i.e. heat) [54]. We now discuss how this is not necessarily the case.

In fact one can erase a qubit system in the angular momentum degree of freedom using a spin bath, where the erasure cost still respects a bound of the form $\alpha^{-1} \log 2$ and there is zero heat cost for the erasure [47]. The difference being that the cost is angular momentum dissipated in the spin bath whose generalised “temperature” is α . Because we are discussing here the cost of erasure in the presence of multiple conserved charges, it is natural to wonder if there is a tradeoff between, say, erasure in energy and erasure in angular momentum. In other words, what form does Landauer's principle take for a *hybrid* process where erasure is carried out using two (or more) conserved quantities?

Let us encode classical bits in the states $|00\rangle$ and $|10\rangle$ in \mathcal{H} . We now develop a one-parameter family of optimal protocols:⁹

1. Let us start with the levels $|00\rangle$ and $|01\rangle$ in a maximally mixed state (which describes the single bit memory that must be erased)

$$\rho_1 = \frac{1}{2} |00\rangle\langle 00| + \frac{1}{2} |10\rangle\langle 10|.$$

Also, notice that because $|00\rangle$ and $|10\rangle$ are degenerate in energy, ρ_1 is initially in thermal equilibrium with the thermal bath.

2. Apply operation Op. 1 and map the state $|01\rangle$ with energy $\bar{\epsilon}$ to a new state (still denoted by $|01\rangle$) with energy $\bar{\epsilon} + \Delta\bar{\epsilon}$. Thermalization with the heat bath (Op. 2) dissipates average energy $p(\bar{\epsilon})\Delta\bar{\epsilon}$ at each step in the thermal bath. Repeating these operations in a sequence of $N \rightarrow \infty$ steps with $\Delta\bar{\epsilon} \rightarrow 0$, the total heat dissipated in rising the level from 0 to ϵ (assuming $\Delta\bar{\epsilon}$ to be sufficiently small to allow us to replace the sum with an integral) is given by

$$\Delta E = \int_0^\epsilon \frac{e^{-\beta\bar{\epsilon}}}{1 + e^{-\beta\bar{\epsilon}}} d\bar{\epsilon} = \frac{1}{\beta} \ln \left(\frac{2}{1 + e^{-\beta\epsilon}} \right).$$

The final state of the memory to be erased is

$$\rho_2 = \frac{1}{1 + e^{-\beta\epsilon}} |00\rangle\langle 00| + \frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}} |10\rangle\langle 10|.$$

⁹ The optimality will be clear from the bound discussed in the next section

3. Now we make use of the charge degree of freedom. The two states $|00\rangle$ and $|01\rangle$ are degenerate in charge Q . Following the same procedure as before we induce a level-raising $|01\rangle$ from 0 to q , where we choose

$$q = \beta\epsilon/\alpha. \quad (19)$$

The reason for this choice will be clear in a moment. This can be done at no cost, because there is no population on $|01\rangle$, i.e. $p(q) = 0$ throughout the process.

4. We can apply a unitary that performs the swap $|00\rangle \rightarrow |00\rangle$, $|01\rangle \leftrightarrow |10\rangle$, $|11\rangle \rightarrow |11\rangle$. This unitary does not respect either the H nor the Q conservation law on the system, and so must utilize the battery systems (a H -battery and a Q -battery). The associated changes in expectation values are

$$\Delta E = -\frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}}\epsilon, \quad \Delta Q = \frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}}q,$$

and the final state, using Eq. (19), is

$$\rho_3 = \frac{1}{1 + e^{-\alpha q}} |00\rangle\langle 00| + \frac{e^{-\alpha q}}{1 + e^{-\alpha q}} |01\rangle\langle 01|.$$

5. We can complete the erasure leaving the system in contact with the Q -bath and rising the level $|01\rangle$ from q to ∞ . Thanks to the choice of the initial q (Eq. (19)), ρ_3 is initially in equilibrium with the Q -bath. Hence this part of the protocol is formally analogous to steps 1-2, but is achieved using a physically different bath (e.g. a spin bath). The charge cost of the partial erasure in the charge basis is then

$$\Delta Q = \int_{\frac{\beta\epsilon}{\alpha}}^{+\infty} \frac{e^{-\alpha q}}{1 + e^{-\alpha q}} dq = \frac{1}{\alpha} \ln(1 + e^{-\beta\epsilon}).$$

The final (erased) state of the memory is $|00\rangle$. This completes the protocol for the single conserved charge.

The total costs in Q and E can be expressed in terms of the single parameter ϵ , the energy at which we decide to swap basis:

$$\Delta E_{tot} = \frac{1}{\beta} \ln\left(\frac{2}{1 + e^{-\beta\epsilon}}\right) - \frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}}\epsilon,$$

$$\Delta Q_{tot} = \frac{1}{\alpha} \ln(1 + e^{-\beta\epsilon}) + \frac{e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}} \frac{\beta}{\alpha} \epsilon.$$

The result is shown in Fig. 3. Each value of ϵ defines a different protocol, corresponding to a point “energy-cost” versus “charge cost” graph. The protocols described yields a curve $(\Delta E_{tot}(\epsilon), \Delta Q_{tot}(\epsilon))$ for fixed temperatures β and α .

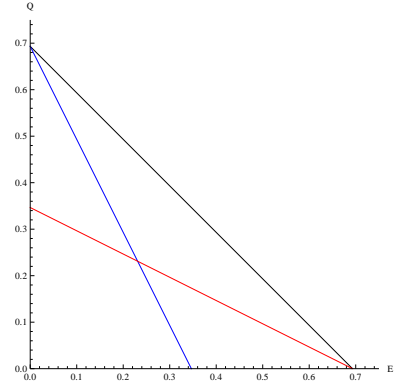


FIG. 3. Energy – charge tradeoff for erasure of a memory through two baths in the protocols provided. The protocols achieve tightly the bound of Eq. (20). Different points along each curve correspond to different swapping points ϵ in the erasure. Black curve: $\beta = \alpha = 1$; red curve: $\beta = 1$, $\alpha = 2$; blue curve: $\beta = 2$, $\alpha = 1$.

As expected, we recover the usual Landauer erasure, $\Delta E_{tot} = \beta^{-1} \ln 2$, in the $\epsilon \rightarrow \infty$ limit and the erasure at no energy, $\Delta Q_{tot} = \alpha^{-1} \ln 2$ of [47], in the $\epsilon \rightarrow 0$ limit. The curves in Fig. 3 interpolating between these two limits will be shown to be tight in the next section. This constitutes a generalization of Landauer principle for multiple observables, and shows how simple trade-offs exist for erasure in the presence of conserved charges. For a fixed set of free states \mathcal{F} , this generalized Landauer principle states that there exists a forbidden region around the origin in the erasure costs graph, as one might expect.

Note that in a real experiment it might not be physically possible to modify the level structure arbitrarily, as was assumed here. This is the case for a Q -bath being a spin bath. One wishes to change the level structure (in angular momentum) by introducing multiple aligned spins and performing CNOT operations [47]. However Nature places a physical lower bound on the discrete steps of \hbar . Naturally, this does not change the general picture presented in the optimal protocol above, but does introduce non-optimality, which might be worth exploring (see Fig. (4)).

We also briefly note that an experiment realising optimal Landauer erasure as described above is sketched in Fig. 5. It could be performed by initiating erasure on an energy qubit defined by a Zeeman splitting; halfway through the erasure protocol, the energy qubit is swapped into a second qubit defined by angular momentum d.o.f. (and screened from the magnetic field inducing the Zeeman splitting, so that it is degenerate in energy). Then by injecting spins and performing CNOT gates, as mentioned above (and in [47]), while equilibrating the system with a spin bath, one completes the erasure. A major challenge for a such protocol is the need for a detailed control of the interaction between the angular momentum qubit and the spin bath (see [47]).

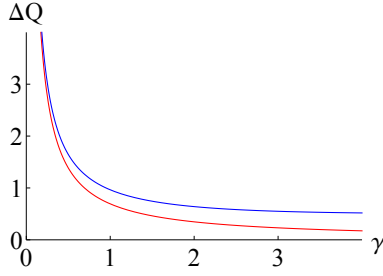


FIG. 4. **Angular momentum baths and \hbar discreteness.** The red curve is the Landauer bound for a single charge, $\Delta Q = \alpha^{-1} \ln 2$. The blue curve represents the erasure cost Q as a function of α when level raising is discrete (e.g., spin bath), by convention $\delta_Q = 1$. For actual experimental scenarios, discreteness is relevant outside the $\alpha \ll 1$ limit. In particular the minimum erasure cost is $\delta_Q/2$ and not 0 for $\alpha \rightarrow \infty$, because erasure requires at least one step to be performed.

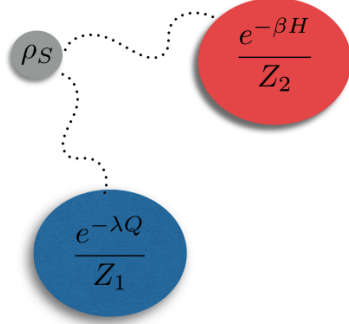


FIG. 5. **Generalised Landauer erasure.** A two level system is erased using both a thermal bath and a spin bath. What is then the general tradeoff between different conserved charges dissipated in the process of erasure?

C. Non-commutative charges

Since we have now an understanding of how Landauer erasure occurs in the presence of two conserved observables we turn to the question of erasure in the presence of non-commutative charges, and determine if there are deviations from the above bounds.

It is possible to place general bounds for Landauer erasure with non-commuting degrees of freedom via a line developed in [55], and compare with the Landauer bounds for the commuting case. Let us consider a memory system S to be erased, together with free states involving non-commuting degrees of freedom. As described, the protocol consists of a sequence of CNOTs between the initial system and a set of ancillas, alternated with thermalisations (unitaries coupling system and bath). Then we can describe the k -th step of the process as a unitary U_k coupling the k -th system $S(k)$, the k -th ancilla and a bath (this step includes a level raising followed by a thermalisation). In the next step, k -th sys-

tem and k -th ancilla are to be considered together as the $(k+1)$ -th system, and a fresh ancilla is introduced. Hence we can define $\rho_{S(k)B} = \rho_{S(k)} \otimes \gamma_B$, where γ_B is a particular bath state (potentially different at every step, but with the same temperatures; however we avoid denoting this for simplicity of the notation). This transforms the state on $S(k)$ into $\rho_{S(k+1)} = \text{Tr}_B[U_k(\rho_S \otimes |0\rangle\langle 0| \otimes \gamma_B)U_k^\dagger]$.

However we are interested in the Landauer bound on heat-flow into the different non-commuting degrees of freedom. To this end, one computes $S(\rho_{S(k+1)B} || \rho_{S(k+1)} \otimes \gamma'_B)$, where $\rho_{S(k+1)B} = U_k(\rho_S \otimes |0\rangle\langle 0| \otimes \gamma_B)U_k^\dagger$. Making the form of γ_B explicit and assuming no erasure in energy, one finds that in a single step we have that

$$-\Delta S_{S(k)} + S(\rho_{S(k+1)B} || \rho_{S(k+1)} \otimes \gamma_B) = \lambda_1 \Delta A_k + \lambda_2 \Delta B_k - S(\gamma'_B || \gamma_B),$$

where $\Delta S_{S(k)} = S(\rho_{S(k+1)}) - S(\rho_{S(k)})$ is the change in von-Neumann entropy of the system and $\gamma'_B = \text{Tr}_{S(k+1)}[\rho_{S(k+1)B}]$ is the final state of the bath. Here $\Delta A_k = \text{Tr}[A(\gamma'_B - \gamma_B)]$, $\Delta B_k = \text{Tr}[B(\gamma'_B - \gamma_B)]$ are the “heat” flows in the non-commuting degrees of freedom, due to the interaction of the system with the free state γ_B . Summing over k and using that both the initial and final states $\rho_{S(0)}$ and $\rho_{S(N)}$ (where N denotes the number of steps in the protocol) are both rank 2 density matrices, we have that $\sum_{k=1}^N -\Delta S_{S(k)} = S(\rho_{S(N)} - S(\rho_{S(0)})) \leq \log 2$. Then, because $S(\rho || \sigma) \geq 0$ with equality if and only if $\rho = \sigma$, we deduce that any non-commutative erasure process is lower bounded by the expression

$$\lambda_1 \Delta A + \lambda_2 \Delta B \geq \ln 2 \quad (20)$$

where $\Delta A := \sum_k \Delta A_k$, and similarly for B . We see that the protocols presented in the commuting case satisfy tightly this bound.

The above analysis shows that the resultant trade-off is not particularly sensitive to the fact that A and B do not commute. The only impact the non-commutativity has is that throughout the sequence of bath interactions one obtains a “time-dependent” pair of expectations $(\langle A(k) \rangle, \langle B(k) \rangle)$, and this pair will be subject to the uncertainty in the two observables. However the same result is obtained for commuting charges, and so we find that non-commuting charges (as defined here) does not produce a particularly different account of erasure. We leave a more detailed analysis to future work.

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